Euler's Theorem for Homogeneous Functions

Let $f : C \to \mathbb{R}$. And f is differentiable in C, and homogeneous of degree m. Then:

$$x_1 \frac{\partial f}{\partial x_1} + x_2 \frac{\partial f}{\partial x_2} + x_3 \frac{\partial f}{\partial x_3} + ... + x_n \frac{\partial f}{\partial x_n} = mf(x)$$

Proof

We define the function g as:

$$g = f(tx_1, tx_2, ..., tx_n)$$

Applying the chain rule:

$$\frac{\partial g}{\partial t} = \frac{\partial f}{\partial x_1} x_1 + \frac{\partial f}{\partial x_2} x_2 + ... + \frac{\partial f}{\partial x_n} x_n$$

Since f is homogeneous of degree n:

$$g = f(tx_1, tx_2, ..., tx_n) = t^m f(x_1, x_2, ..., x_n)$$

Therefore:

$$\frac{\partial g}{\partial t} = mt^{m-1} f(x_1, x_2, ..., x_n)$$

With the expression from before we have:

$$\frac{\partial f}{\partial x_1} x_1 + \frac{\partial f}{\partial x_2} x_2 + ... + \frac{\partial f}{\partial x_n} x_n = mt^{m-1} f(x_1, x_2, ..., x_n)$$

And if we take $t = 1$:

$$\frac{\partial f}{\partial x_1} x_1 + \frac{\partial f}{\partial x_2} x_2 + ... + \frac{\partial f}{\partial x_n} x_n = mf(x_1, x_2, ..., x_n)$$